Parametrization-Independent Non-Uniform Fourier Approach to Path Synthesis of Four-Bar Mechanism

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Abstract: This paper deals with the classical problem of dimensional synthesis of planar four-bar linkages for path generation. Using Fourier descriptors, a given path is represented by harmonic series. Extensive research has been done on this approach in mechanism synthesis with the assumption that the path is assigned a prescribed parametrization or timing beforehand, and the input link of mechanisms rotates with constant angular velocity. However, little research efforts have been put into pure path synthesis independent of parametrization. In this paper, we present an exact method that can efficiently and accurately carry out the pure path matching using arc length parametrization. Meanwhile, curve normalization combined with artificial neural network is used to decouple the design space, which leads naturally to a fast synthesis approach.

Keywords: Linkage synthesis, Fourier descriptors, Parametrization

I. Introduction

This paper studies the problem of dimensional synthesis of planar four-bar linkages for path generation using Fourier descriptors. Central to this problem is the formulation of an error function that quantifies the deviation of the generated path from the desired path in terms of nine independent variables associated with the design of a four-bar linkage. When the error function attempts to use all nine variables to compare the shape, size, location, and orientation of the two paths simultaneously, it makes the resulting optimization routine highly inefficient.

Synthesis approach that has received increasing attention lately is the use of Fourier descriptors for linkage synthesis. This idea was first explored by Freudenstein [1] in the context of function generation. The research was followed by Funabashi [2], Farhang et al. [3], [4], Chu and Cao [5], and McGarva [6], [7], and Ullah and Kota [8], and Nie and Krovi [9]. Lately, Chu and Sun [10], [11], [12] have extended Fourier descriptor based method to the synthesis of spherical and spatial linkages. One of the key features of the Fourier descriptor based method is the ability to decouple the nine design variables involved in path generation. Ullah and Kota [8] was the first to present this conclusion and used it to reduce the dimension of the search space from nine to five. Recently, Wu et al. [13] further reduced the search dimension from five to four.

Another important feature is that while the path of a coupler point depends on the choice of the coupler point, one may extract a subset of Fourier descriptors of the path in such a way that they depend only on the linkage dimensions but not the choice of the coupler point. This means that for each four bar linkage, one set of Fourier descriptors can be used to tag all its coupler curves. Chu and Wang [14] made this key observation and achieved significant reduction in the size of the database for numerical atlas. Xie and Chen [15] was the first to extend Fourier descriptor method to the image space of kinematic mapping to solve the whole cycle motion generation problem in four bar linkage synthesis. In their work, image curve of a desired motion was indexed with Fourier descriptors (FDs) which were used to be matched with those of four-bar coupler motion. Neural networks was used to establish the relationship between FDs and dimensions of a four-bar linkage.

While the Fourier-based approach has been extensively used to synthesize mechanism for path and motion generation, it has its own limitation in such application: dependency on parametrization or timing. Nie and Krovi [9] noticed this problem and artfully utilized it to render smallest number of harmonic components for synthesizing coupled multi-link serial open chain with smallest number of links. Vasiliu and Yannou [16] also mentioned parametrization issue in their paper but in effect handled the problem of different samplings under a given parametrization, which is a sampling-independent method. In general, Fourier transform is conducted against timing parameter t in signal processing. However, problem arises when it comes to geometry analysis. When processed by Fourier Transform, different parametric forms of a task curve would yield different Fourier descriptors. In Fig. 1, the unit circle is assigned two different parametrizations $z_1(t)$ and $z_2(t)$. The Fourier transform of $z_1(t)$ is $z_1(t) = e^{i2\pi t}$ while $z_2(t)$ is decomposed as:

$$z_2(t) = (0.1607 + 0.3138i) + (-0.0759 + 0.0285i) e^{-i\pi t} + (0.6117 - 0.6020i) e^{i\pi t} + (0.0053 - 0.0018i) e^{-i\pi t} + (0.3111 + 0.1758i) e^{i\pi t} + (-0.0045 + 0.0019i) e^{-i\pi t} + (-0.0090 + 0.0674i) e^{i\pi t}$$

It is clear that two sets of Fourier descriptors are completely different from each other, though they both define the same geometric curve, i.e., the unit circle.

As with task curve, the mechanism coupler curve shares the same problem. When the crank rotates, a curve would be traced out by coupler point of the mechanism. For a mechanism of fixed dimensions, different crank rotation functions could lead to different parametrization or timing for the coupler curve. Traditional ways of path synthesis approaches (see Chu [14] and Wu [17]) assume that crank always rotates with constant angular velocity and hence

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Arc length parametrization is a parametrization based on the inherent property of curve: arc length. When comparing two curves by their Fourier descriptors, we could reparametrize both of them using arc length as parameter and then initiate the comparison. However, direct arc length parametrization for coupleur curve of four-bar mechanism requires determination of ten design variables simultaneously, which would incur tremendous computational cost. In Wu’s [17] and Chu’s [14] method, the ten-design-parameter synthesis problem is converted to a four-design-parameter problem because of the assumption that crank rotates constantly. In our case, we need to find a different way to reduce the design cost. Considering that a curve has position, orientation, size and shape, we can first match the shape of task and four-bar coupleur curve by curve normalization [18]. The process of curve normalization takes out information about curve’s position, orientation and size and merely keep the shape of curve. In later section, it will be shown that the shape of four-bar coupleur curve only depends on three link ratios and the choice of coupleur point. Therefore, we try to match Fourier descriptors of four-bar coupleur curve after curve normalization with that of task curve after curve normalization, both under arc length parametrization, so as to find link ratios and the choice of coupleur point. Because we record the position, orientation and size of the task curve along its normalization process, those data are used to find the other five design variables for four-bar mechanism. In order to efficiently determine link ratios and choice of coupleur point, an Artificial Neural Network (ANN) is trained to establish the relationship between them and the Fourier descriptors of normalized coupleur curve under arc length parametrization.

When conducting Fourier transform against curve under arc length parametrization, the arc length parameters associated with sampling points could be unevenly distributed. Therefore, Non-uniform Discrete Fourier Transform (NDFT) [19] should be adopted instead of regular DFT. There are two forms of non-uniform Discrete Fourier transform: the samples are irregularly taken in the time domain t but irregularly taken in the frequency domain; the samples are irregularly taken in the time domain t but regularly taken in the frequency domain. We take the latter one in the sense that DFT is a special case of the second NDFT. As said earlier, the curve is given with a certain parametrization and a uniformly sampled parameter t; after reparametrization with arc length, the arc length parameter would be irregularly spaced, which leads to the use of NDFT.

The organization of the paper is as follows. Section 2 reviews how Fourier descriptors is used for representing a closed curve on the plane as well as the concept of NDFT. Section 3 presents the loop closure equation of a four bar linkage in a form that is suitable for the development of this paper and its Fourier analysis. Section 4 introduces how to decouple design variables with curve normalization. Section 5 details on the numerical analysis of arc-length parametrization, artificial neural networks and direct search method. Section 6 gives results and discussion of our approach to justify its accuracy in pure path generation, with comparison against conventional Fourier-based path generation algorithms.

II. Fourier Descriptor Based Path Representation

This section reviews Fourier descriptors (FDs) in the context of characterizing a path generated by a point on a rigid body.

A planar rigid body is shown in in Fig. 2. The position of the moving body relative to a fixed frame F is represented by a frame M attached to the moving body. Assume that the rigid body is moving in the plane, one point v traces out the closed curve (task curve) marked with dashed line while the one with solid line is matching curve to be synthesized by four-bar mechanism. In elementary differential geometry [20], the task curve is called plane Peano curve in the sense that it is a continuous map of the unit interval [0,1] into the plane. However, there could be an infinity of maps in terms of a plane Peano curve, which signifies that a curve would have numerous parametric forms. In other words, a curve can have different parametrizations or timings against parameter t, t ∈ [0,1].

![Fig. 1. Solid line: parametrization 1. Dash line: parametrization 2. a) Shape of unit circle in parametrization 1; b) Shape of unit circle in parametrization 2; c) x component of parametrization function z(t) under two different parametrizations; d) y component of parametrization function z(t) under two different parametrizations.](image)

![Fig. 2. F is the fixed frame and M the moving frame. Dashed-line path is generated by the point v on the rigid body and solid-line path synthesized by linkage.](image)
When task curve is represented by a sequence of points rather than a continuous curve, Discrete Fourier Transform (DFT) is used to characterize the curve. For a certain parametrization of task curve, there could be different sampling ways of parameter \( t \). Usually, \( t \) is uniformly sampled between \([0, 1]\) with \( N \) knots. Given with a certain parametric function \( z(t) = x(t) + jy(t) \) of the task curve, its inverse DFT can be represented as

\[
Z \left( \frac{n}{N} \right) = \frac{1}{N} \sum_{k=0}^{N-1} Z_k e^{j2\pi nk/N} \quad 0 \leq n \leq N - 1 \tag{2}
\]

where the Fourier coefficients are given by forward DFT

\[
Z_k = \sum_{n=0}^{N-1} z \left( \frac{n}{N} \right) e^{-j2\pi nk/N} \tag{3}
\]

However, in practice, we want to take into consideration a more general case, which is the case where the samples are irregularly taken in the time domain but regularly taken in the frequency domain. Then, non-uniform Discrete Fourier Series (NDFT) comes to play. The definition of the forward NDFT is defined as follows

\[
Z_k = \sum_{n=0}^{N-1} z \left( \frac{t_n}{T} \right) e^{-j2\pi nt/T} \tag{4}
\]

where \( T \) is the range of extension for the samples \( t_n \) and \( t_n \) is irregularly sampled. For our plane Peano curve, \( T \) is taken to be 1s. DFT is a special case of NDFT if NDFT is required to uniformly distributed in frequency domain and thereby it would be appropriate to use NDFT whether the sampling is uniform or not.

For Fourier approximation of curve, we only need a small number of harmonic components. Therefore, the inverse NDFT can be combined with finite Fourier series (FFT) and reformulated as

\[
z \left( \frac{t_n}{T} \right) = \sum_{k=-p}^{p} Z_k e^{j2\pi knt/T} \quad 0 \leq t \leq 1 \tag{5}
\]

where \( \omega_0 = \frac{2\pi}{T} \) and \( p \) is a small positive integer that defines the maximum order of the harmonic terms used in the approximation. Such a curve is also known as a trigonometric polynomial curve of order \( p \) in the field of computer aided geometric design. In this paper, we follow the least square approach proposed by Wu [17] to find the Fourier coefficients \( Z_k \) in that this approach is able to deal with both DFT and NDFT.

III. Fourier Analysis of the Four-Bar Mechanism

Consider a planar four-bar linkage shown in Fig. 3 with \( XOY \) being the fixed coordinate frame. The fixed pivot \( A_0 \) is located at point \((x_0, y_0)\) with \( AB_0 \) being the ground link and \( A_0A \) the input link. Let \( l \) denote the length of the \( i \)th link and and \( \theta \) the angle measured from the \( x \) axis of the fixed frame. Let \( \phi, \lambda \) and \( \psi \) be the angles of link \( A_0A, AB, B_0B \) as measured from the ground link \( A_0B_0 \), respectively.

Assume that the input link rotates with angular velocity \( \omega(t) \), we have

\[
\phi = \omega(t) + \phi_0 \quad t \in [0, 1] \tag{6}
\]

where \( \phi_0 \) is the initial input angle.

Using loop closure equations, it has been shown in [17] that the coupler angle \( \lambda \) is given by

\[
e^{j\lambda} = -\frac{B(\phi) \pm \sqrt{B(\phi)D_2(\phi)}}{2A(\phi)} \tag{7}
\]

where

\[
l_{21} = l_2/l_1, \quad l_{31} = l_3/l_1, \quad l_{41} = l_4/l_1 \tag{8}
\]

\[
A(\phi) = l_{11}(l_{21} e^{-j\phi} - 1) \tag{9}
\]

\[
B(\phi) = 1 + l_{31} - (l_{31} + l_{41})^2 - 2l_{21} \cos \phi \tag{10}
\]

\[
\Delta_1(\phi) = 1 + l_{31}^2 - (l_{31} + l_{41})^2 - 2l_{21} \cos \phi \tag{11}
\]

\[
\Delta_2(\phi) = 1 + l_{31}^2 - (l_{31} + l_{41})^2 - 2l_{21} \cos \phi \tag{12}
\]

and the sign \( \pm \) correspond to the two configurations of the four-bar linkage for each input angle.

In view of Eq. (7), it is clear that the coupler angle \( \lambda \) depends only on the input angle \( \phi \) as well as link ratios

\[
E(\phi) \Delta_1(\phi) \Delta_2(\phi) \leq 0 \tag{13}
\]

The input link is crank if this inequality holds for all \( \phi \in [0, 2\pi] \); otherwise, it is a rocker.

Now let us consider Fourier representation of the coupler curve of a four-bar mechanism. Let \( D_0 = x_0 + jy_0 \) be the complex pivot specifying the fixed pivot \( A_0 \) and let \( z = r e^{i\theta} \) represent the position \( P \) with respect to the coupler link \( AB \). The position of the coupler point relative to global frame \( XOY \) can be represented as

\[
P = A_0 + l_1 e^{i\phi_1} + z e^{i\phi} = A_0 + l_2 e^{i\theta_1} e^{i\phi} + z e^{i\theta} e^{j\lambda} \tag{14}
\]

Ten design variables \( \{l_1, l_2, l_3, l_4, x_0, y_0, \theta_1, \theta, r, \alpha, \phi_0\} \) are included in the above equation. Also, \( P \) can be described by Fourier series as

\[
P = \sum_{k=-\infty}^{\infty} \beta_k e^{ik2\pi t} \quad t \in [0, 1] \tag{15}
\]

For the task curve, its Fourier descriptors, say \( \alpha_k \), are obtained under arc length parametrization. Next, we would like to match \( \alpha_k \) with coupler curve FDs \( \beta_k \). Therefore, we
have to simultaneously search foregoing ten design variables, compute \( P \) based on Eq. (14) and obtain \( \beta_6 \) from Eq. (15) by NDFT to see whether \( \alpha_6 = \beta_6 \). Obviously, such brutal-force searching would take impractical amount of time. For the sake of minimizing synthesis cost, we introduce curve normalization in next section to efficiently minimize the search cost by decoupling design variables.

IV. Decoupling of Design Variables

In this section, we will detail on curve normalization. Ten design variables \( \{l_1, l_2, l_3, l_4, x_0, y_0, \theta_3, r, \alpha, \phi_0\} \) are here noted through previous discussion. By looking at those design variables, we find that only \( l_1, l_2, l_3, l_4, r \) and \( \alpha \) determine the shape and size of coupler curve. \( x_0 \) and \( y_0 \) play a role in translating the coupler curve as a whole; \( \theta_3 \) rotates the coupler curve as whole; \( \phi_0 \) only decides the starting point of coupler curve, which has nothing to do with shape and size. Another reason to ignore \( \phi_0 \) is that Fourier transform have a property of shift-invariance, i.e., it does not depend on the starting point. Therefore, Eq. (14) can be split into two parts

\[
P = P_1 + P_2 e^{i\theta_3}
\]

where \( P_1 = A_0 = x_0 + y_0 \) and \( P_2 = l_0 e^{i\phi_0} + z e^{i\beta} \). \( P_1 \) dictates the position of the coupler curve in that it translates the whole curve by a vector of \( (x_0, y_0) \). \( \phi_0 \) in \( P_2 \) can be chosen according to user needs since it only relates to the starting point of a coupler curve. \( P_2 \) represents the shape and actual size of coupler curve. Furthermore, we can divide \( P_2 \) by \( l_1 \) to get \( \tilde{P}_2 = l_0/l_1 e^{i\phi_0} + z/l_1 e^{i\beta} \), which still keeps the shape of coupler curve. Then \( z \) is expressed as \( x_c + y_c i \) instead of \( r e^{i\alpha} \). Finally, in view of Eq. (7), the shape of coupler curve is determined by five design variables \( \{l_{21}, l_{31}, l_{41}, x_c/l_2, y_c/l_3\} \). For clear representation of coupler point coordinate \( (x_c, y_c) \) on coupler link \( l_1 \), \( x_c/l_1 \) and \( y_c/l_1 \) are changed to \( x_c/l_1 \) to \( y_c/l_1 \) respectively and five design variables become

\[
\{l_{21}, l_{31}, l_{41}, x_c/l_1, y_c/l_1\}
\]

(17)

The advantage of this process is that the space of design variables is reduced to five and the shape of the coupler curve kept. However, the position, orientation and size of the coupler curve are altered. Before comparing it with the task curve, curve normalization is required to transform both task curve and the new coupler curve \( \tilde{P}_2 \) into their canonical configuration. After normalization, both curves would be limited to the bounding unit rectangle whereby pure shape match can be implemented.

Curve normalization procedure (Fig.4) was first proposed by Dikabar and Mruthyunjaya [18] and later used by Sánchez Marín and Pérez González [21]. Recently, Galán-Marín et al. [22] applied it with wavelet descriptors approach to efficiently synthesize crank-rocker mechanism constrained by optimal transmission angle. Principle of this procedure is that we treat the closed curve along which its mass is uniformly distributed. Through curve normalization, the major principal axis of moment of inertia is aligned with the x-axis; then, the width \( w \) and height \( h \) of the bounding box of the curve are evaluated; finally, the curve with the bounding box is scaled by a factor of \( 1/w \) and translated to the origin.

Though the process of curve normalization changes the position, orientation and size of curve, it holds the shape unchanged. As mentioned previously, shape exclusively depends on \( \{l_{21}, l_{31}, l_{41}, x_c/l_1, y_c/l_1\} \). As long as these parameters are known, we can compare task curve with four-bar curve in their canonical configurations, thus efficiently reducing the search space of design variables.

V. Numerical Synthesis

In this part, we explain how to find the five design variables in (17). Our method is approximating the relationship between Fourier descriptors of curve and those five design parameters. The Fourier descriptors come from the curve after being normalized and reparametrized using arc length as parameter. For the purpose of better accuracy, we add direct search approach as the post-processing stage for neural network to obtain final values of five design variables. Finally, we will use a restoring approach, which restores the task curve to its original position, orientation and size, to find the rest of four design variables.

A. Arc length parametrization

As said earlier, a curve can have distinct parametrizations, which are called various representatives of the curve. We want to single out a unique representative of the curve in a geometrically significant way. This is done by referring a curve to its arc length as a parameter. A curve \( x(\mu) \) is said to be defined as a function of its arc length if the tangent vector \( x'(\mu) = dx/d\mu \) is a unit vector. \( |x'(\mu)| = 1 \). Then, \( \mu \) becomes the parameter of arc length \( s \).

Now consider the task curve with its canonical configuration and assume the parametrization is given as continuous function of \( z(t) = x(t)+y(t) \). Theoretically, the arc length parametrization can be computed with following steps: a) by \( s(t) = \int_{t_0}^{t} \|z'(\mu)\|d\mu \), we get the arc length function \( s \) against \( t \); b) compute \( s^{-1}(t) \), the inverse function of \( s(t) \), and we get time \( t \) function \( t(s) = s^{-1}(t) \) against arc length; c) substitute \( t(s) \) into \( z(t) \) and finally the arc length...
parametrization \( z(s) \) is obtained. However, it’s impossible to derive an explicit formula for \( s^{-1}(t) \).

Nonetheless, \( z(t) \) is usually given as a sequence of \( N \) points and thereby numerical approach can be used. Assume we have the sequence of \( z(t) \) sampled as \( z_0, z_1, \ldots, z_{N-1} \). Next, we treat the curve as polygonal curve and compute the arc length as follows

\[
s(n) = \begin{cases} 
0, & n=0 \\
\sum_{k=1}^{n} \left\| z\left( \frac{k}{N} \right) - z\left( \frac{k-1}{N} \right) \right\|, & n=1, \ldots, N-1
\end{cases}
\]

For the sequence of \( z(n/N) \) \((n = 0, 1, \ldots, N-1)\), we obtain a corresponding sequence of \( s(n) \) by Eq.(18). Therefore, we formulate the arc length parametrization \( z_\alpha(s) = z_\alpha(s(n)) \), \( 0 \leq s(n) \leq L \) \((L \text{ is the total length of curve})\). Applying DFT to \( z_\alpha(s) \) requires that the domain of \( s(n) \) be \([0,1]\). So we normalize \( s(n) \) by a factor of \( 1/L \). One thing worth attention here is that even though \( n/N \) is uniformly spaced between 0 and 1, there is no guarantee of uniformity for \( s(n)/L \). Hence, NIDFT instead of DFT should be employed.

**B. Artificial Neural Network**

Inspired by biological neural networks, an artificial neural network is a computational structure consisting of a collection of interconnected elements, known as neurons, to define a function [23], [24]. The network function is largely determined by the nature of the connections, which can be adjusted to map an input space to the corresponding desired output space.

Neuron is the elemental component of an artificial neural network (Fig. 5). The core of a neuron is a transfer function \( F \) which maps the sum of the weighted input \( w^T l \) and the bias \( b \) to the output, i.e.,

\[
c = F(w^T l + b)
\]

where \( l \) denotes the vector of all the inputs \( I_i \) to the neuron, and \( w \) denotes the vector of all the weights \( W_i \) of the connections between the inputs and the neuron. A neural network is formed by layers of neurons, where the outputs of one layer become the inputs of next layer. A typical neural network architecture is shown in Fig 6. The weights between the \( kth \) and \((k+1)th\) layers are defined in a weight matrix \( W^{k+1,k} \). Consequently, the output vector of the \((k+1)th\) layer becomes

\[
e^{k+1} = P^{k+1} (W^{k+1,k}e^k + b^{k+1})
\]

In this paper, we use the widely-adopted backpropagation (BP) algorithm to train a network to fit the input-output relationship embedded in the sample data. BP is a supervised learning method which fits a function based on samples of input-output data pairs. For each input vector, the algorithm estimates the error between the actual and desired network outputs, and backpropagates it from the output layer to hidden neurons to estimate the contribution of each hidden neuron to the output error. It calculates the gradient of each weight, which indicates the direction of error increase, and updates the weight in the opposite direction of the gradient.

After curve normalization and reparametrization using arc length, we have the Fourier coefficients \( a_k, k = -p, \ldots, p \). From the work of Li et al. [25], it is known that for any four-bar mechanism with an input crank, magnitudes of Fourier coefficients decrease noticeably as the order \( p \) goes higher. So we choose coefficients of order from \(-3 \) to \( 3 \), i.e., \( a_3, a_{-3}, \ldots, a_2, a_{-2}, a_1 \) to be Fourier descriptors of a curve. In order to train the neural network, we produced 101700 sets of Fourier descriptors of different four-bar curves generated by varying design variables in (17) to train the neural network while using the other 101700 sets to test the validity of our trained neural network to see whether it can effectively approximate the relationship between harmonic components and design variables in (17).

In practical design, the ratio between any two links is not expected to be extremely large or small. So it is reasonable to predefine a max link ratio in the design process, \( K_{\text{max}} \). Correspondingly, the minimum link ratio is \( 1/K_{\text{max}} \). Therefore, we vary three link ratios \( l_{21}, l_{31}, l_{41} \) in \([1/K_{\text{max}}, K_{\text{max}}]\). In this paper, \( K_{\text{max}} \) is taken to be 6. Likewise, it is required that the coordinate of coupler point on coupler link needs to be dimensionally compatible with the length of coupler link. So we have similar constraints for \( x_c/l_1 \) and \( y_c/l_3 \) whose \( \xi_{\text{max}} \) is 3.

**C. Restoring method**

At this moment, we know the values of five design variables \( l_{21}, l_{31}, l_{41}, x_c/l_1, y_c/l_1 \). From the discussion of section 4, \( \overline{P}_2 \) can be computed. According to Eq. (16), our goal is to find the four-bar curve \( P \) that best matches the task curve, say, \( T \). Hence, we let \( T = P \) and (16) becomes:

\[
T = P_1 + P_2 e^{i\theta_1}
\]

Next, there are three steps to take in order to match the size, orientation and position of \( T \) and \( P \), during which \( x_0, y_0, \theta_1 \), \( l_2 \) can be found:

1. Size match between \( T \) and \( P \). Rotate \( T \) and \( P \) to align their major principal axis with \( x \)-axis of fixed frame.
respectively and name the transformed curves as $R(T)$ and $R(\tilde{P}_2)$. Compute the width or height of the bounding boxes of $R(T)$ and $R(\tilde{P}_2)$ respectively and denote the ratio as $\frac{w_1}{w_2}$ or $\frac{h_1}{h_2}$, which is the size ratio between $T$ and $\tilde{P}_2$. According to Eq. (16), size of $P$ is determined by $P_2$ and $P_2$ is equal to $l_1\tilde{P}_2$. Therefore, $l_1 = \frac{w_1}{w_2} = \frac{h_1}{h_2}$.

2. Orientation match between $T$ and $P$. We obtain the value of $P_2$ at step 1 by $l_1\tilde{P}_2$. According to Eq. (16), the orientation difference between $P$ and $P_2$ lies on $\theta_1$. Therefore, $\theta_1$ is measured as the angle from the major principal axis of $P_2$ to that of $T$.

3. Position match between $T$ and $P$. Until now, we know $P_2e^{i\theta_1}$. Then, we compute the center for $T$ and $P_2e^{i\theta_1}$ correspondingly and denote them as $C_1$ and $C_2$. The distance vector from $C_1$ to $C_2$ equals to $l_1$.

Up until now, values of nine design variables in (16) have been found except for $\phi_0$. As said earlier, $\phi_0$ only determines the starting point of curve and is irrelevant to position, orientation, size and shape of curve. In practical use, the starting point can be chosen according to the needs of users. The flowcharts of synthesis algorithm are shown in Figs. 7 and 8.

VI. Examples

In this section, we present some examples to show the effectiveness of our approach and compare with Wu’s and Chu’s methods respectively. First, we present a task curve that is generated by coupler point $P$ of the Watt II six-bar shown in Fig. 9. Four-bar $AEFG$ functioning as driver linkage is serially chained with four-bar $ABCD$ of which $P$ serves as the coupler point to trace out a coupler curve. $AEFG$ must be a double-crank mechanism since both $GF$ and $AE$ should be able to rotate in full circle respectively. When $GF$ rotates with constant angular velocity, $AE$ usually rotates with varying angular velocity. Hence, three different timings are produced by altering the lengths of $GF$, $FE$ and $AG$ to change the rotating pattern of $AE$. Link $AB$, $BC$, $CD$, $BP$ and $CP$ remain the same to keep the closed curve traced by $P$ unchanged under these three timings. Nine design parameters are chosen as: $x_0=2.2$, $y_0=3.5$, $l_1=4.4$, $l_{121}=0.5$, $l_{12}=2.8$, $l_{13}=2.7$, $x_c=1.0$, $y_c=0.8$, $\theta_1=0.55$. Three sets of $AE$, $EF$, $FG$ and $AG$ are given as follows:

- **Parametrization I**: $AE = 0.50$, $EF = 1.60$  (22)
  $FG = 1.60$, $AG = 0.35$

- **Parametrization II**: $AE = 0.50$, $EF = 0.80$  (23)
  $FG = 0.75$, $AG = 0.40$

- **Parametrization III**: $AE = 0.50$, $EF = 2.35$  (24)
  $FG = 2.40$, $AG = 0.10$

Rotation functions of link $AB$ corresponding to each parametrization are presented in Fig. 10. By observing the figure, we can see that parametrization III is the closest to perfect timing in which link $AB$ rotates uniformly while parametrization I and parametrization II deviate from perfect timing, indicating that link $AB$ rotates with fluctuating angular velocity.

For these different parametrizations, their corresponding sets of Fourier descriptors are listed in Table I. It is clear from the table that different parametrizations yield distinct sets of Fourier descriptors even though they relate to the same curve. After arc length parametrization imposed on each case, we obtain three new sets of Fouriers in Table II.

By observing the data in Table II, three sets of Fourier descriptors are almost same just as we discussed in previous chapters. Moreover, for the next two examples, we compare with traditional synthesis approaches. In order to compare their methods with ours against the same timing, their
results are presented under arc length parametrization.

**A. Comparison against Wu’s Approach**

First presented are the results of our method. In Table III, design parameters corresponding to three parametrizations are displayed. We can see that those parameters are close to those of the four-bar mechanism presented at the beginning that generates the task curve. In Table IV, Fourier descriptors for three parametrizations are shown.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_3$</td>
<td>0.9743</td>
<td>1.0661</td>
<td>0.6969</td>
</tr>
<tr>
<td>$C_2$</td>
<td>1.7290</td>
<td>1.5949</td>
<td>1.4765</td>
</tr>
<tr>
<td>$C_1$</td>
<td>2.3956</td>
<td>2.0666</td>
<td>3.2828</td>
</tr>
<tr>
<td>$C_0$</td>
<td>16.7382</td>
<td>17.2055</td>
<td>17.0292</td>
</tr>
<tr>
<td>$C_1$</td>
<td>1.7154</td>
<td>1.4730</td>
<td>2.3377</td>
</tr>
<tr>
<td>$C_2$</td>
<td>1.1184</td>
<td>1.0410</td>
<td>0.7418</td>
</tr>
<tr>
<td>$C_3$</td>
<td>0.4841</td>
<td>0.5770</td>
<td>0.2318</td>
</tr>
</tbody>
</table>

**TABLE I. Magnitudes of Fourier descriptors of the task curve under three different parametrizations**

Second, we use Wu’s synthesis approach [17] to obtain three four-bar mechanisms and show results in Tables V and VI. According to Table V, design parameters are quite different from the task four-bar mechanism. Also from Table VI and Table II, difference in Fourier components is notable. The graphical comparisons of our method and Wu’s are demonstrated in Figs. 11, 12 and 13. Among three parametrizations, parametrization III is the closest to perfect timing as pointed out earlier and therefore Wu’s method can yield good match with original curve. The other two parametrizations twist perfect timing to the extent that distinctly reduces the exactness of Wu’s method.

**B. Comparison against Chu’s Approach**

In this example, we would compare our method with Chu’s one in [14], another widely used Fourier-based synthesis approach. The task curve and three parametrizations are same as those specified previously and therefore results of our method are identical to those in Tables III and IV.

Then, we use Chu’s approach to obtain three four-bar mechanisms and show results in Tables VII and VIII. The graphical comparisons of our method and Chu’s are revealed in Figs. 11, 12 and 13. By comparing Chu’s results with those from Wu’s in Example 1, we find that both approaches can output good match when the parametrization is nearly perfect, as in the case of parametrization III. However, the more the parametrization varies from perfect one, the more their synthesized curves deviate from the task curve, which is justified by comparing synthesized curves of both methods under parametrization I with those of parametrization II in Figs. 11 and 12. Clearly, the results for parametrization I are better than those of II because parametrization I is closer to perfect timing than II as shown in Fig. 10.

**TABLE III. The design parameters of synthesized four-bar linkages under three parametrizations by our method**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_2$</td>
<td>0.5045</td>
<td>0.5072</td>
<td>0.5077</td>
</tr>
<tr>
<td>$l_3$</td>
<td>2.8020</td>
<td>2.8039</td>
<td>2.7963</td>
</tr>
<tr>
<td>$l_4$</td>
<td>2.7070</td>
<td>2.7096</td>
<td>2.7080</td>
</tr>
<tr>
<td>$x_c$</td>
<td>1.0028</td>
<td>0.9942</td>
<td>1.0011</td>
</tr>
<tr>
<td>$y_c$</td>
<td>0.8075</td>
<td>0.7908</td>
<td>0.8036</td>
</tr>
<tr>
<td>$l_1$</td>
<td>4.3286</td>
<td>4.3549</td>
<td>4.3228</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.5495</td>
<td>0.5615</td>
<td>0.5599</td>
</tr>
<tr>
<td>$x_0$</td>
<td>2.3715</td>
<td>2.4686</td>
<td>2.4719</td>
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<tr>
<td>$y_0$</td>
<td>3.5416</td>
<td>3.4464</td>
<td>3.5058</td>
</tr>
</tbody>
</table>

**TABLE IV. Magnitudes of Fourier descriptors (after arc length parametrization) of the task curve under three different parametrizations**
TABLE V. The design parameters of synthesized four-bar linkages under three parametrizations by Wu’s method.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_{21})</td>
<td>0.7980</td>
<td>0.8554</td>
<td>0.6052</td>
</tr>
<tr>
<td>(l_{31})</td>
<td>1.7515</td>
<td>4.9052</td>
<td>1.8951</td>
</tr>
<tr>
<td>(l_{41})</td>
<td>1.9051</td>
<td>4.9980</td>
<td>1.8050</td>
</tr>
<tr>
<td>(x_c)</td>
<td>0.8903</td>
<td>0.2190</td>
<td>1.3764</td>
</tr>
<tr>
<td>(y_c)</td>
<td>0.6551</td>
<td>0.4433</td>
<td>0.6998</td>
</tr>
<tr>
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<td>3.5239</td>
<td>4.3040</td>
</tr>
<tr>
<td>(x_0)</td>
<td>10.4813</td>
<td>10.8015</td>
<td>5.4999</td>
</tr>
<tr>
<td>(y_0)</td>
<td>0.7741</td>
<td>3.4920</td>
<td>2.1181</td>
</tr>
</tbody>
</table>

TABLE VI. Magnitudes of Fourier descriptors of three synthesized coupler curves by Wu’s method.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_3)</td>
<td>0.3588</td>
<td>0.3148</td>
<td>0.3954</td>
</tr>
<tr>
<td>(C_2)</td>
<td>0.2429</td>
<td>0.2303</td>
<td>0.2058</td>
</tr>
<tr>
<td>(C_1)</td>
<td>3.8635</td>
<td>3.8858</td>
<td>3.7170</td>
</tr>
<tr>
<td>(C_0)</td>
<td>18.1680</td>
<td>18.6918</td>
<td>17.8296</td>
</tr>
<tr>
<td>(C_1)</td>
<td>1.5478</td>
<td>1.4226</td>
<td>1.8367</td>
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<tr>
<td>(C_2)</td>
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</tr>
<tr>
<td>(C_3)</td>
<td>0.1569</td>
<td>0.1279</td>
<td>0.1532</td>
</tr>
</tbody>
</table>

Fig. 11. Comparison of Curves under parametrization I

Fig. 12. Comparison of Curves under parametrization II

Fig. 13. Comparison of Curves under parametrization III

TABLE VII. The design parameters of synthesized four-bar linkages under three parametrizations by Chu’s method.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_{21})</td>
<td>0.8051</td>
<td>0.9011</td>
<td>0.6050</td>
</tr>
<tr>
<td>(l_{31})</td>
<td>1.8950</td>
<td>4.9550</td>
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<td>(l_{41})</td>
<td>2.0522</td>
<td>5.0032</td>
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<tr>
<td>(x_c)</td>
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<td>0.2112</td>
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<tr>
<td>(y_c)</td>
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<td>(l_1)</td>
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<td>(y_0)</td>
<td>0.6750</td>
<td>3.5521</td>
<td>1.1477</td>
</tr>
</tbody>
</table>

TABLE VIII. Magnitudes of Fourier descriptors of three synthesized coupler curves by Chu’s method.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_3)</td>
<td>0.3704</td>
<td>0.3994</td>
<td>0.3939</td>
</tr>
<tr>
<td>(C_2)</td>
<td>0.2667</td>
<td>0.2338</td>
<td>0.2020</td>
</tr>
<tr>
<td>(C_1)</td>
<td>3.7364</td>
<td>3.8833</td>
<td>3.6882</td>
</tr>
<tr>
<td>(C_0)</td>
<td>18.1640</td>
<td>18.7623</td>
<td>17.8883</td>
</tr>
<tr>
<td>(C_1)</td>
<td>1.5323</td>
<td>1.4473</td>
<td>1.8343</td>
</tr>
<tr>
<td>(C_2)</td>
<td>0.3102</td>
<td>0.3250</td>
<td>0.2736</td>
</tr>
<tr>
<td>(C_3)</td>
<td>0.1415</td>
<td>0.0523</td>
<td>0.1512</td>
</tr>
</tbody>
</table>

VII. Conclusions

In this paper, we apply the technique of arc length parametrization to eliminate the side effect of parametrization (timing) in traditional Fourier-based path synthesis algorithms and implement the pure shape match. By employing curve normalization, we manage to reduce the search space of design parameters. Moreover, with the help of Neural Network, five design parameters can be quickly determined once the network is trained and rest of four parameters are solved by restoring method, specifically by comparing size, orientation and position with task curve. The results are compared with Wu’s and Chu’s methods to show that our method is independent of parametrization inherent in the path.

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References


