Abstract: Parallel manipulators (PMs) with decoupled kinematics can be obtained by combining a translational PM (TPM) with a spherical PM (SPM) either in multiplatform architectures or in integrated more-complex architectures. Some of the latter type are inspired by the 6-4 fully parallel manipulator (6-4 FPM), whereas others of the same type are deduced by suitably combining TPMs’ limbs and SPMs’ limbs into more cumbersome limbs which contain more than one actuated joint. The decoupled PMs (DPMs) presented here pursue an intermediate concept between the last two, which keeps all the actuators on or near to the base in a simplified architecture with only three limbs. These features preserve the lightness of the mobile masses, together with the associated good dynamic performances, and reduce the limitations on the workspace due to limb interferences. The type synthesis of these DPMs is addressed till to identify sixty-three architectures; then, the position analysis problems of the identified architectures are solved in closed form.

Keywords: Parallel manipulators, decoupled kinematics, position analysis, type synthesis

1 Introduction

The possibility of decoupling position and orientation in end effector’s pose control is an appealing feature. It allows the use of simplified path-planning algorithms which sequentially involve a limited number of actuators and make the machine behave in an easy-to-visualize manner that facilitates the operator. This decoupling is easy to implement in serial manipulators [1], whereas it requires a specific design in parallel manipulators (PMs) [2 – 11] due to their intrinsically coupled kinematics.

PMs with decoupled kinematics can be obtained by combining in series a translational PM (TPM) and a spherical PM (SPM) into a multiplatform architecture [5]. Such simple solutions, in general, lose many of the advantages of parallel architectures. Combining a TPM and an SPM so that the TPM controls the position of the spherical motion center of the SPM [6] is a better idea whose most known implementation is the 6-4 fully parallel manipulator (6-4 FPM) [2]. This concept generates many decoupled PM (DPM) architectures since many are the TPM and SPM architectures proposed in the literature [12]. The so-deduced DPM architectures feature six limbs with one actuator per limb (three of the TPM and three of the SPM) which still is somehow cumbersome.

An alternative concept relies on the identification of suitable limb types which give the sought-after decoupling when put together [3, 10, 11]. This technique has the advantage of identifying architectures with a reduced number of limbs (usually three [11]), but requires more than one actuator per limb.

Here, a family of DPM architectures with three limbs is identified and studied. Such architectures feature all the actuators on or near to the base. This property makes them save all the advantages of 6-degrees-of-freedom (6-dof) PMs with six limbs and one actuator per limb. Also, their three-limbed architecture makes their workspace wider since the limb interferences and the constraints on the workspace [13] are reduced; whereas, their decoupled position analysis makes their control easier.

This paper is organized as follows. Section 2 presents the novel DPM architectures. Section 3 addresses their position analysis. Eventually, section 4 draws the conclusions.

2 Type Synthesis

Parallel manipulators (PMs) feature two rigid bodies, one fixed (base) and the other mobile (platform) connected to one another through a number of kinematic chains (limbs). Hereafter, the notation \( g_1 \text{string}_1-g_2 \text{string}_2-\ldots-g_n \text{string}_n \), where \( g_i \) for \( i=1,\ldots,n \) are integers, denotes a structure or a manipulator constituted by two rigid bodies, named base and platform, joined through \( (g_1+g_2+\ldots+g_n) \) kinematic chains (limbs) whose topologies are given by the strings separated with hyphens. The i-th string, \( \text{string}_i \), lists the sequence of kinematic pairs encountered when moving from the base to the platform and the number \( g_i \) indicates how many limbs with that topology are present; \( g_i \) is omitted if it is equal to one. Capital letters indicate the type of kinematic pair in these strings: S, P, R and U stand for spherical, prismatic, revolute pairs and universal joint, respectively, and an underlined capital letter indicates an actuated pair when the notation refers to manipulators’ architectures; also, the symbols || or \( \perp \) between two R indicate that the axes of the two adjacent R pairs are parallel or perpendicular to one another, respectively. Eventually, the phrase “limb connectivity” indicates the degrees of freedom of the relative motion between platform and base in the kinematic chain constituted by platform and base uniquely connected through the limb whose connectivity [14 – 16] is considered.

Figure 1 shows an S-RS-US structure. With reference to Fig. 1, \( O \) is the center of the S pair that directly joins platform and base. \( B_3 \) and \( A_3 \) are the S-pair center and the foot of the perpendicular passing through \( B_1 \) on the revolute-pair axis, respectively, in the RS limb; whereas, \( A_2 \) and \( B_2 \) are the centers of the U joint of and of the S pair, respectively, in the US limb.
The proposed family of DPMs is constituted by all the three-limbed architectures with actuators on or near to the base that generate an S-RS-US structure (Fig. 1) when the actuators are locked. The deduction of these architectures moves from the fact that “each limb of a six-dof PM has to guarantee six degrees of freedom to the relative motion between platform and base (i.e., the limb connectivity must be six)”. Henceforth, only six-dof limbs (i.e., non-redundant limbs) will be considered. This assumption and the previous consideration bring to state that, in the sought-after architectures,

(i) the S limb of the S-RS-US structure must be generated by a limb of $S$S type where the platform is joined through an S pair to one link of a spatial three-dof mechanism, generically denoted $S$, with three actuated pairs,

(ii) the RS limb of the S-RS-US structure must be generated by a limb either of $S$S-RY type or of $S$S-XY type or of $S$S-RY type where X and Y are any two single-dof pairs/mechanisms, and

(iii) the US limb of the S-RS-US structure must be generated by a limb either of $S$S-US type or of $S$S-US type where Z is a single-dof pair/mechanism.

Therefore, the architectures of this family must belong to one out of the six “generic” types: $S$S-XYS-ZUS, $S$S-XYS-ZUS, $S$S-XYS-ZUS, $S$S-XYS-ZUS, $S$S-XYS-ZUS, and $S$S-XYS-ZUS. Each particular architecture is fully defined when the topologies of the $S$S, X, Y and Z pairs/mechanisms are defined. These topologies are particularized by taking into account the other two structural conditions that identify the family, that is, (a) the manipulator must be a DPM and (b) the actuated pairs must be on or near to the base. An actuated $P$ pair can be considered “near to the base” even though it is adjacent to the $S$ pair that joins the limb to the platform since $UP$ or $PS$ limbs are commonly adopted in fully-parallel manipulators [17]; whereas, an actuated $R$ pair can be considered “near to the base” if it is adjacent to another $R$ pair that joins the limb to the base since, in this case, it can be easily actuated by keeping the actuator on the base.

The generic $S$S limb is sufficient to satisfy condition (a) since the spatial 3-dof mechanism, $S$, is able to fully control the position of one platform point (point O of Fig. 1) with its three actuated pairs no matter how the other two limbs are devised. The topology of the $S$S mechanism is limited by condition (b) and the fact that it has to constitute a unique limb when combined with the $S$ pair. $S$S limbs that satisfy these constraints are the RRPS limb (Fig. 2(a)) and either of the two single-loop $R$P$R$ limb and $R$P$R$ limb shown in Figs. 2(b) and 2(c), respectively. The $S$S mechanisms of the $R$P$R$ limb and $R$P$R$ limb are both five-bar planar linkages whose motion plane is hinged to the base through an actuated $R$ pair. Even though other solutions (e.g., the PPPS or RRPPS limbs) are possible, only these three $S$S limbs will be considered here.

Regarding the remaining two limbs, if X, Y and Z are either $R$ or $P$ pairs and the limbs that do not satisfy condition (b) are excluded, the following limb types are identified:

- Limbs of $X$S type: $R$P$S$, $P$R$P$,
- Limbs of $Y$S type: $R$S$R$, $R$P$S$, $P$P$S$;
- Limbs of $Y$S type: $R$P$S$, $P$R$S$;
- Limbs of $Z$S type: $U$S$P$.

The combination of the above-identified limb types yields 63 ($=3\times7\times3$) DPM architectures. One out of these DPM architectures is the RRPS-RRPPS-UPS (Fig. 3) which deserves to be highlighted for its symmetry since it features three limbs with the same RRPS topology that are actuated in different ways.
3 Position Analysis

PMs’ position analysis refers to the solution of two finite-kinematics problems: the direct position analysis (DPA) and the inverse position analysis (IPA). The DPA is the determination of the positions of the three S-pair centers (i.e., points O, B1, and B2 in Fig. 1) and the IPA of each limb yields the actuated-joint variables that are compatible with an assigned position of their ending S-pair center.

The DPA of all the above-identified DPM architectures can be solved with the same algorithm since their DPA solutions one-to-one correspond to the assembly modes of the S-RS-US structure generated when the actuators are locked. Instead, their IPA depends on the adopted limb types. Actually, the assigned platform pose gives the positions of the three S-pair centers (i.e., points O, B1, and B2 in Fig. 1) and the IPA of each limb yields the actuator assembly modes of the S-RS-US structure generated when the triangle B1OA1, d1 and d2 are the lengths of the segments A1B1 and A2B2, respectively. The angle θ is a geometric constant of the RS limb that depends on the actuated-joint variables of the kinematic chain that generated it; whereas, the angle θ is the joint variable of the R pair of the RS limb; eventually, θ is an angle whose value, together with the above-defined geometric parameters, uniquely identifies the assembly mode of the S-RS-US structure. In the RRPS-RRPS-UPS architecture (Fig. 3), d1 and d2 are actuated-joint variables and, if u3 is chosen parallel the axis of the actuated R pair of the RRPS limb, θ is an actuated-joint variable, too.

The following analytic relationships hold among the above-defined geometric parameters, (all the vectors are measured in a reference system embedded in the base):

\[ \mathbf{u}_2 = \mathbf{u}_0 \cos \theta + (\mathbf{u}_1 \times \mathbf{u}_0) \sin \theta, \quad \mathbf{u}_2 \times \mathbf{u}_3 = (\mathbf{u}_0 \times \mathbf{u}_1) \cos \theta + \mathbf{u}_0 \sin \theta \]

(1a)

\[ \mathbf{B}_1 - \mathbf{A}_1 = d_1 [(\mathbf{u}_1 \cos \theta + (\mathbf{u}_1 \times \mathbf{u}_0) \sin \theta)] \]

(1b)

\[ \mathbf{u}_3 = \frac{\mathbf{B}_1 - \mathbf{O}}{b_1} = \frac{d_1 [\mathbf{u}_1 \cos \theta + (\mathbf{u}_1 \times \mathbf{u}_3) \sin \theta]}{b_1} \]

(1c)

\[ \sin \delta = \sqrt{1 - \left(\frac{b_1^2 + a_1^2 - d_1^2}{2a_1b_1}\right)^2}, \quad j = 1,2 \]

(1d)

\[ \mathbf{u}_4 = (\mathbf{A}_1 - \mathbf{O}) - \mathbf{u}_3 [(\mathbf{A}_1 - \mathbf{O}) \cdot \mathbf{u}_3] \]

(1e)

\[ \mathbf{u}_3 \times \mathbf{u}_4 = \frac{\mathbf{u}_3 \times (\mathbf{A}_1 - \mathbf{O})}{a_1 \sin \delta} \]

\[ = \frac{d_1 [\mathbf{u}_1 \cos \theta + (\mathbf{u}_1 \times \mathbf{u}_3) \sin \theta] \times (\mathbf{A}_1 - \mathbf{O})}{a_1 b_1 \sin \delta} \]

(1f)

\[ \mathbf{B}_2 - \mathbf{O} = (\mathbf{B}_2 - \mathbf{A}_2) + (\mathbf{A}_2 - \mathbf{O}) = b_2 [\mathbf{u}_4 \cos \beta + \sin \beta (\mathbf{u}_3 \cos \theta + (\mathbf{u}_1 \times \mathbf{u}_3) \sin \theta)] \]

(1g)

where \( \mathbf{u}_0, \mathbf{u}_1, (\mathbf{A}_1 - \mathbf{O}) \) and \( (\mathbf{A}_2 - \mathbf{O}) \) are known constant vectors and \( \theta \) is a known constant angle.
Relationships (1) when introduced into the closure equations (see Fig. 4)

\[
b_1^2 = (B_1-O)(B_1-O)[(B_1-A_1) + (A_1-O)] \\
(2a)
\]

\[
d_2^2 = (B_2-A_2)(B_2-A_2)[(B_2-O) - (A_2-O)] [(B_2-O) - (A_2-O)] \\
(2b)
\]

yield the following two equations in explicit form

\[
a_1^2 - b_1^2 + d_1^2 + 2d_1(A_1-O)(u_1 \cos \theta_2 + (u_2 \times u_3) \sin \theta_2) = 0 \\
(3a)
\]

\[
a_2^2 + b_2^2 - d_2^2 - 2b_2(A_2-O) \cdot [u_3 \cos \beta + \sin \beta (u_1 \cos \theta_2 + (u_2 \times u_3) \sin \theta_2)] = 0 \\
(3b)
\]

Equations (3) constitute a system of two trigonometric equations in the two unknowns \( \theta_2 \) and \( \theta_0 \). Actually, Eq. (3a) contains only \( \theta_2 \); thus, it can be immediately solved by transforming it into a quadratic equation in \( \tan(\theta_2/2) \) through the two trigonometric identities \( \cos \theta_2 = \frac{1 - \tan^2(\theta_2/2)}{1 + \tan^2(\theta_2/2)} \) and \( \sin \theta_2 = 2 \tan(\theta_2/2) / (1 + \tan^2(\theta_2/2)) \) (see Appendix A). Such equation gives at most two values for \( \theta_2 \) that geometrically correspond to the two possible intersections (see Fig. 4) between the sphere with center \( O \) and radius \( b_1 \) and the circumference with center \( A_1 \) and radius \( d_1 \) that lies on a plane perpendicular to \( u_2 \).

If the value of \( \theta_2 \) is known, Eq. (3b) becomes a trigonometric equation in \( \theta_0 \) which can be immediately solved by transforming it into a quadratic equation in \( \tan(\theta_0/2) \) through the two trigonometric identities \( \cos \theta_0 = \frac{1 - \tan^2(\theta_0/2)}{1 + \tan^2(\theta_0/2)} \) and \( \sin \theta_0 = 2 \tan(\theta_0/2) / (1 + \tan^2(\theta_0/2)) \) (see Appendix A). Therefore, at most two values of \( \theta_0 \) can be computed through Eq. (3b) for each value of \( \theta_2 \) computed through Eq. (3a). These two values of \( \theta_0 \) geometrically correspond to the two possible intersections between the sphere with center \( A_2 \) and radius \( d_2 \) and the circumference with center on \( OB_1 \) and radius \( b_2 \sin \beta \) that lies on a plane perpendicular to \( u_3 \).

The conclusion is that there are at most four assembly modes of a general S-RS-US structure and as many are the DPA solutions of all the studied DPMs.

### 3.2 Inverse Position Analysis

Here, the positions of the three spherical-pair centers (i.e., points \( O, B_1 \) and \( B_2 \)) in Fig. 3) are known since the platform pose is assigned and the values of the actuated-joint variables must be computed. Each of these centers is the endpoint of a limb and its position is the input datum of this limb’s IPA. Thus, each limb’s IPA can be solved independently from the others and, then, the IPA solutions of the analyzed DPM are obtained by combining the IPA solutions of each limb.

The IPAs of all the serial 6-dof kinematic chains (limbs) with an ending S pair and containing only R and/or P pairs have been solved in closed form in [1] where the demonstration that at most four limb configurations are compatible with an assigned position of the S-pair center is provided. Moreover, in [18] (as reported in [19], pp. 160-161), Takano showed that, with a general layout, (i) at most four solutions are found for limbs containing either three R pairs, or two R pairs and one P pair, whereas (ii) at most two solutions are found for limbs containing one R pair and two P pairs. Condition (i) occurs for four (i.e., RRPS, RRRS, RPRS, RRPS) out of the above-listed seven kinematic-chain types that generate an RS limb, when the actuators are locked, and condition (ii) occurs for the remaining three (i.e., PRPS, PRPS, PPRPS).

The number of IPA solutions can be further reduced if the limbs satisfy special geometric conditions. This is the case of the three types of kinematic chain which generate a US limb when the actuators are locked. In particular, the UPS limb yields only one solution for the distance \( d_2 \) (i.e., the only actuated-joint variable of this limb) between \( A_2 \) and \( B_2 \) (see Fig. 4) since \( A_2 \) is a fixed point of the base. Instead, the RUS and PUS limbs yield two solutions for the actuated-joint variable. In fact, in these two cases, \( d_2 \) is a fixed distance, and \( A_2 \) must simultaneously lie on the sphere with center \( B_2 \) and radius \( d_2 \) and on either a circumference, in the case of the RUS limb, or a line, in the case of the PUS limb, both fixed to the base; thus, \( A_2 \) may be located at either of the at-most-two intersections between a sphere and a circumference or a line.

Figure 5: Kinematic schemes of the three limbs of $(3W)S$ type shown in Fig. 2: (a) scheme of the RRPS limb, (b) scheme of the single-loop \( R \text{L}(2R)\text{S} \) limb, and (c) scheme of the single-loop \( R \text{L}(2\text{RP})\text{S} \) limb.
it is the length of the segment $A_0O$ and $A_0$ is a fixed point of the base; moreover, the relationship (see Fig. 5a)

\[ v_2 = v_0 \cos \varphi_1 + (v_1 \times v_0) \sin \varphi_1 = \frac{v_1 \times (O - A_0)}{|v_1 \times (O - A_0)|} \]  (4)

makes it possible to uniquely determine $\varphi_1$ and $\varphi_2$ as follows:

\[ \varphi_1 = \text{atan2} \left( \frac{v_1 \times (O - A_0)}{|v_1 \times (O - A_0)|}, \frac{v_1 \times (O - A_0) \cdot v_0}{|v_1 \times (O - A_0)|} \right) \]  (5a)

\[ \varphi_2 = \arcsin \left( \frac{v_1 \times (O - A_0)}{d_0} \right) \]  (5b)

where $\varphi_2 \in [0, \pi]$ rad.

Regarding the two single-loop limbs that have been proposed above to control the platform position [i.e., the $R_{\perp}(2R|R)$S limb (Figs. 2b and 5b) and the $R_{\perp}(2R|S)$ limb (Figs. 2c and 5c)], the actuated-joint variable, $\varphi_1$, in Figs. 5b and 5c, of the actuated $R$ pair that joins the two limb types to the base and identifies the plane of the triangle $A_0A_3$ is uniquely determined when the position of point $O$ is assigned, since points $A_0$ and $A_3$ are fixed point of the base and the relationship (see Figs. 5b and 5c)

\[ v_2 = v_0 \cos \varphi_1 + (v_1 \times v_0) \sin \varphi_1 = \frac{(A_3 - O) \times (A_0 - O)}{|(A_3 - O) \times (A_0 - O)|} \]  (6)

makes it possible to uniquely determine $\varphi_1$ as follows

\[ \varphi_1 = \text{atan2}(v_2 \cdot (v_1 \times v_0), v_2 \cdot v_0) \]  (7)

where $v_2$ is computed through the last expression of Eq. (6).

Moreover, the lengths, $d_0$ and $d_3$, of the segments $A_0O$ and $A_3O$ (see Fig. 5c), which are the remaining two actuated-joint variables of the $R_{\perp}(2R|S)$ limb (Fig. 2c), are uniquely determined, too, since points $A_0$ and $A_3$ are fixed point of the base. Thus, the IPA of the $R_{\perp}(2R|S)$ limb has only one solution.

Eventually, both the angles, $\varphi_2$ and $\varphi_3$ of Fig. 5b, that are the remaining two actuated-joint variables of the $R_{\perp}(2R|S)$ limb (Fig. 2b) have a double determination for an assigned position of point $O$, since each point $C_i$ for $i=0,3$ (see Fig. 5b) may be located at either of the two intersections of the two circumferences, one with center $O$ and radius $d_0$ and the other with center $A_i$ and radius $g_i$, that lie on the plane of the triangle $A_0A_3$. Thus, the IPA of the $R_{\perp}(2R|S)$ limb has at most four solutions.

In short, the number of IPA solutions of the above-identified 63 DPM architectures varies from one to thirty-two ($=4 \times 2 \times 4$) according to the chosen limb types. It is worth stressing that the RRPS-RRPS-UPS architecture shown in Fig. 3 has only one IPA solution.

## 4 Conclusions

Manipulators with fully or partially decoupled kinematics allow the use of simplified path-planning algorithms which sequentially involve a limited number of actuators and make the machine behave in an easy-to-visualize manner that facilitates the operator.

A novel family of decoupled parallel manipulators has been identified which contains sixty-three DPM architectures. The DPMs of this family feature only three limbs with connectivity six and all the actuators on or near to the base. These features make all the members of this family as fast as fully-parallel manipulators with complex kinematics without the use of cumbersome six-limbed architectures.

The position analysis of all the proposed DPMs has been solved through a unified approach and the demonstration that both the direct and the inverse position analyses of these DPMs are easily solvable in closed form has been provided.

Eventually, one DPM architecture of the proposed family has been found which exhibits three limbs with the same topology and a much simple kinematics.

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### Appendix A

The following type of trigonometric equation

\[ h_2 \cos x + h_1 \sin x + h_0 = 0, \]  (A1)

where the coefficients $h_i$ for $i=0, 1, 2$ are real constants and $x$ is the unknown, can be transformed into a quadratic algebraic equation in $t=\tan(x/2)$ by introducing the trigonometric identities

\[ \cos x = \frac{1 - t^2}{1 + t^2}, \quad \sin x = \frac{2t}{1 + t^2} \]  (A2)

and, then, rationalizing the resulting expression.

This algebraic manipulation transforms Eq. (A1) as follows

\[ (h_0 - h_2) t^2 + 2 h_1 t + (h_0 + h_2) = 0 \]  (A3)

Equation (A3) yields the following two explicit-form solutions

\[ t = \frac{-h_1 \pm \sqrt{h_1^2 + h_2^2 - h_0^2}}{h_0 - h_2} \]  (A4)

### References
